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LAYER POTENTIALS FOR ELASTOSTATICS AND HYDROSTATICS IN CURVILINEAR POLYGONAL DOMAINS

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ABSTRACT. The symbolic calculus of pseudodifferential operators of Mellin type is applied to study layer potentials on a plane domain Ω^+ whose boundary $\partial\Omega^+$ is a curvilinear polygon. A "singularity type" is a zero of the determinant of the matrix of symbols of the Mellin operators and can be used to calculate the "bad values" of p for which the system is not Fredholm on $L^p(\partial\Omega^+)$.

Using the method of layer potentials we study the singularity types of the system of elastostatics

$$L\mathbf{u} = \mu\Delta\mathbf{u} + (\lambda + \mu)\nabla \operatorname{div} \mathbf{u} = 0.$$

in a plane domain Ω^+ whose boundary $\partial\Omega^+$ is a curvilinear polygon. Here $\mu > 0$ and $-\mu \leq \lambda \leq +\infty$. When $\lambda = +\infty$, the system is the Stokes system of hydrostatics. For the traction double layer potential, we show that all singularity types in the strip $0 < \operatorname{Re} z < 1$ lie in the interval $(\frac{1}{2}, 1)$ so that the system of integral equations is a Fredholm operator of index 0 on $L^p(\partial\Omega^+)$ for all p, $2 \leq p < \infty$. The explicit dependence of the singularity types on λ and the interior angles θ of $\partial\Omega^+$ is calculated; the singularity type of each corner is independent of λ iff the corner is nonconvex.

INTRODUCTION

Recently there has been considerable interest in using layer potentials to solve L^p boundary value problems for elliptic operators and systems on a Lipschitz domain Ω^+ in $\mathbf{R}^n$. For the systems of elastostatics [DKV] and hydrostatics [FKV], Dahlberg, Fabes, Kenig, and Verchota have used Rellich type identities to prove that the double layer potential integral equations yield a Fredholm operator of index 0 on $L^2(\partial\Omega^+)$. For $p \neq 2$ only limited information is available on the boundary integral equations for general Lipschitz domains in $\mathbf{R}^n$. The general problem of the notion of *symbol* on the boundary of a general Lipschitz domain is still very much open.

In this paper we treat a very special case: a curvilinear polygonal domain in $\mathbf{R}^2$. In this 2-dimensional case a precise symbolic calculus of pseudodifferential operators of Mellin type is available. We show that certain double layer boundary integral equations yield operators which for all p, $2 \leq p < \infty$, are

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Fredholm operators of index 0 on $L^p(\partial\Omega^+)$. The singularities exhibited for $p < 2$ show the limitations of the general theory.

We develop the theory of double layer potentials for treating boundary value problems for second order elliptic systems in a plane domain Ω^+ which is bounded by a curvilinear polygon $\partial\Omega^+$. The double layer potential operators on $L^p(\partial\Omega^+)$ are interpreted as systems of pseudodifferential operators of Mellin type, or more simply *Mellin operators*, on $L^p(0, 1)$. A symbolic calculus for Mellin operators was developed by Lewis and Parenti [LP] and J. Elschner [E]. Our particular interest is to explicitly calculate the singularity types. A *singularity type* of a system of Mellin operators $\mathbf{K}$ is defined as a complex number z_0, $\operatorname{Re} z_0 = \frac{1}{p}$, at which the determinant of the principal symbol, $\operatorname{Smbl}^{\frac{1}{p}}(\mathbf{K})$, vanishes. Elschner [E] has used singularity types to construct parametrices and develop asymptotic expansions for solutions of the equation $\mathbf{Kf} = \mathbf{g}$. For a different approach to a symbol map on curves with corners, see Costabel [C].

In §1 we describe the algebra of Mellin operators on the finite interval $J \equiv [0, 1]$. We follow closely the notation of [E] since the parametrices have meromorphic symbols with poles at the singularity types.

In §2 we describe a class of double layer kernel operators and show that they are examples of Mellin operators; their principal symbols are calculated.

§3 gives a parametrization of a curvilinear polygon $\partial\Omega^+$ which reduces a system of double layer potential integral operators on $L^p(\partial\Omega^+)$ to a big system of operators of Mellin type on $L^p(J)$. The part of the symbol arising from each vertex P_k of $\partial\Omega^+$ is the same as for the corresponding operator in a plane sector of interior opening θ_k. Theorem 2 shows that the "bad values" of p for which the operators are not Fredholm on $L^p(\partial\Omega^+)$ are the same as for the sector problems; for the "good values" of p, the index of the system on $L^p(\partial\Omega^+)$ can be calculated from the change in argument of the principal symbol for the sector problems and Theorem 1 yields the index. Theorem 2 should be considered as a *localization* result.

In §4 we apply our results to for the system of linear elastostatics:

$$L\mathbf{u} = \mu\Delta\mathbf{u} + (\lambda + \mu)\nabla \operatorname{div} \mathbf{u} = 0. \tag{0–1}$$

The numbers μ and λ are the Lamé moduli; we assume $\mu > 0$ and that $-\mu \leq \lambda \leq +\infty$. When $\lambda = -\mu$, the operator L is two copies of the Laplace operator; when $\lambda = +\infty$, we interpret the operator as the Stokes system of hydrostatics:

$$\begin{cases} L(\mathbf{u}, p) = \mu\Delta\mathbf{u} - \nabla p = 0, \\ \operatorname{div} \mathbf{u} = 0. \end{cases} \tag{0–2}$$

Our interest is in the description of the singularities of solutions in terms of the interior angles θ at the vertices of $\partial\Omega^+$ and the parameter λ. We state our results in terms of the normalized parameter b, defined as

$$b = \frac{\lambda + \mu}{\lambda + 2\mu}, \tag{0–3}$$

so that $0 \leq b \leq 1$.

The boundary operator of physical significance is the traction operator. The *stress tensor* $\mathbf{T} = (T_{i,k})$ is defined by

$$T_{i,k}(\mathbf{u}) = \lambda(\operatorname{div}\mathbf{u})\delta_{i,k} + \mu(u_{i,k} + u_{k,i}), \tag{0–6}$$

or in the case of the Stokes system ($\lambda = +\infty$),

$$T_{i,k}(\mathbf{u}, p) = -p(x)\delta_{i,k} + \mu(u_{i,k} + u_{k,i}), \tag{0–7}$$

where $u_{i,k} = \partial u_i/\partial x_k$. If $\vec{\nu}$ is the outward normal to Ω^+ at a point $P \in \partial\Omega^+$, the traction operator is

$$\mathbf{T}_{\vec{\nu}}(\mathbf{u}) = \mathbf{T}(\mathbf{u})\vec{\nu}. \tag{0–8}$$

We shall also consider another conormal boundary operator

$$\mathbf{N}_{\vec{\nu}}(\mathbf{u}) = \mu\frac{\partial \mathbf{u}}{\partial \vec{\nu}} + (\lambda + \mu)(\operatorname{div}\mathbf{u})\vec{\nu}, \tag{0–9}$$

which for $b = 0$ reduces to the Neumann boundary operator. Let Ω^- denote the complement of $\Omega^+ \cup \partial\Omega^+$. The boundary value problems we shall treat are

(1) The Dirichlet problems $D_\pm$:

$$\begin{cases} L\mathbf{u} = 0 \quad \text{in } \Omega^\pm, \\ \mathbf{u}|_{\partial\Omega^\pm} = \mathbf{g} \in L^p(\partial\Omega^+). \end{cases} \tag{0–10}$$

(2) The traction problems $T_\pm$:

$$\begin{cases} L\mathbf{u} = 0 \quad \text{in } \Omega^\pm, \\ \mathbf{T}_{\vec{\nu}}(\mathbf{u})|_{\partial\Omega^\pm} = \mathbf{g} \in L^p(\partial\Omega^+). \end{cases} \tag{0–11}$$

(3) The Neumann problems $N_\pm$:

$$\begin{cases} L\mathbf{u} = 0 \quad \text{in } \Omega^\pm, \\ \mathbf{N}_{\vec{\nu}}(\mathbf{u})|_{\partial\Omega^\pm} = \mathbf{g} \in L^p(\partial\Omega^+). \end{cases} \tag{0–12}$$

We represent the solutions of $D_\pm$ as double layer potentials and the solutions of $T_\pm$ and $N_\pm$ as single layer potentials using the fundamental solution given by Kupradze [K, Chapter 9, (9.2)]:

$$\mathbf{\Gamma}(X) = (\Gamma_{i,j}(X)) = \left(\delta_{i,j}\frac{n}{2\pi}\log r^2 - \frac{m}{\pi}\frac{x_i x_j}{r^2}\right), \tag{0–13}$$

with $r^2 = x_1^2 + x_2^2$ and

$$n = \frac{\lambda + 3\mu}{2\mu(\lambda + 2\mu)}, \quad m = \frac{\lambda + \mu}{2\mu(\lambda + 2\mu)}. \tag{0–14}$$

This fundamental solution satisfies

$$L(\mathbf{\Gamma}(X)) = 2\,\delta(X)\,\mathbf{I}, \tag{0–15}$$

where the operator L is applied to the columns of the matrix $\mathbf{\Gamma}$. When $b = 1$, we have $n = m$ and as in Ladyzhenskaya [La, Chapter 3] introduce the fundamental pressure (row) vector:

$$\mathbf{q}(X) = \frac{1}{\pi}\frac{X}{r^2},$$

so that $\{\mathbf{\Gamma}, \mathbf{q}\}$ is a solution of the adjoint Stokes system

$$\text{(0–16)} \qquad \begin{cases} \mu\Delta\mathbf{\Gamma} + \nabla\mathbf{q} = 2\,\delta(X)\,\mathbf{I}, \\ \operatorname{div}\mathbf{\Gamma} = 0. \end{cases}$$

The solution of $D_{\pm}$ is sought in the form of the double layer potential

$$\text{(0–17)}^1 \qquad \mathbf{u}_{\mathbf{T}}(X) = \int_{\partial\Omega^+} \mathbf{T}_{\vec{\nu}(Q)}(\mathbf{\Gamma}(X-Q))\,\mathbf{f}(Q)\,d\sigma_Q.$$

Taking nontangential limits in $L^p(\partial\Omega^+)$ from inside and outside Ω^+, and calling the resulting limits $\mathbf{u}_{\mathbf{T}}^{\pm}$, we obtain

$$\text{(0–18)} \qquad \mathbf{u}_{\mathbf{T}}^{\pm}(P) \equiv \mathbf{K}_{\mathbf{T}}^{\pm}\mathbf{f}(P) = \pm\mathbf{I}\mathbf{f}(P) + \text{p.v.}\int_{\partial\Omega^+} \mathbf{T}_{\vec{\nu}(Q)}(\mathbf{\Gamma}(P-Q))\mathbf{f}(Q)\,d\sigma_Q,$$

where even in the case where $\partial\Omega^+$ is flat the integral operator in (0–18) is not compact.

In a like manner the solutions of $T_{\pm}$ and $N_{\pm}$ are represented in the form of a single layer potential

$$\text{(0–19)} \qquad \mathbf{u}_{\mathbf{S}}(X) = -\int_{\partial\Omega^+} \mathbf{\Gamma}(X-Q)\mathbf{f}(Q)\,d\sigma_Q.$$

Applying the boundary operators $\mathbf{T}_{\pm}$ and $\mathbf{N}_{\pm}$ to $\mathbf{u}_{\mathbf{S}}$ we obtain integral equations which are adjoints to the double layer integral equations; e.g.,

$$[\mathbf{T}_{\pm}(\mathbf{u}_{\mathbf{S}})\vec{\nu}](P) = (\mathbf{K}_{\mathbf{T}}^{\mp})^*\mathbf{f}(P).$$

In §4 we give explicit expressions for the kernels for elastostatics and hydrostatics in a plane sector.

In §5 we compute the symbols for the problems in a plane sector. Theorem 7 gives a very simple expression for the determinant of the matrix of symbols in terms of the parameter b and the interior angle θ.

In §6, we calculate the singularity types of $\mathbf{K}_{\mathbf{T}}^{\pm}$. We first summarize the results in a a plane sector in Theorem 8. Theorem 8 shows that there is a contrast in the cases of a corner of Ω^+ where Ω^+ is convex ($0 < \theta < \pi$), and the case of a reentrant corner ($\pi < \theta < 2\pi$). We first note that when $b = 0$, the operator $\mathbf{T}_{\vec{\nu}}$ does not cover L; however, $\mathbf{N}_{\vec{\nu}}$ covers L for $0 \le b \le 1$. The nature of the singularity types is

[1] In the case $b = 1$, the kernel $\mathbf{T}_{\vec{\nu}(Q)}(\mathbf{\Gamma}(X-Q))$ is replaced by

$$\mathbf{T}'_{\vec{\nu}(Q)}(\mathbf{\Gamma}(X-Q), \mathbf{q}) \equiv (\mathbf{q}\,\delta_{i,k} + \mu(\Gamma_{i,k} + \Gamma_{k,i}))\vec{\nu}(Q),$$

the stress tensor being applied to the columns of $\{\mathbf{\Gamma}, \mathbf{q}\}$.

Case I. For $0 < \theta < \pi$, the Mellin operators $\mathbf{K}_{\mathbf{T}}^{+}$ and $\mathbf{K}_{\mathbf{N}}^{+}$ have the same singularities for $0 < b \leq 1$. For $0 < b < 1$, there are two singularity types in the strip $0 < \operatorname{Re} z < 1$; both singularity types are real and lie in $(\frac{1}{2}, 1)$. When $b = 1$, there is a value $\gamma_{\text{crit}} \approx 257°27'$ for which there are two singularity types for $0 < \theta < 2\pi - \gamma_{\text{crit}}$; for $2\pi - \gamma_{\text{crit}} \leq \theta < \pi$, there is only one singularity type in the strip.

Case II. For $\pi < \theta < 2\pi$, the singularity types for $\mathbf{K}_{\mathbf{T}}^{+}$ in the strip $0 < \operatorname{Re} z < 1$ are independent of b, lie in $(\frac{1}{2}, 1)$ and approach $\frac{1}{2}$ as θ approaches 2π; there is one singularity type in the strip for $\pi < \theta \leq \gamma_{\text{crit}}$; a second singularity type develops for $\gamma_{\text{crit}} < \theta < 2\pi$.

Finally, Theorem 9 summarizes the "good values" and "bad values" of p for the double layer potential integral equations on $L^p(\partial\Omega^+)$, where $\partial\Omega^+$ is a curvilinear polygon.

1. Mellin operators on a finite interval

Algebras of Mellin operators on $J \equiv [0, 1]$ are defined in [LP, Definition (4.1)] and [E, Definition (4.1)]. We follow closely the notions of [E] since Elschner develops an extension to meromorphic symbols which arise in constructing parametrices. For $0 \leq \alpha < \beta \leq 1$, define the strip $\Gamma_{\alpha,\beta} = \{z \in \mathbf{C} : \alpha < \operatorname{Re} z < \beta\}$, and let Γ_γ be the line $\{z = \gamma + i\xi : -\infty \leq \xi \leq +\infty\}$. The symbol space $\tilde{\Sigma}^0_{\alpha,\beta}$ is defined in [E, Definition (1.12)].

For $f \in C_0^\infty(\mathbf{R}^+)$ define the *Mellin transform* of f by

(1–1) $$\mathscr{M} f(z) = \tilde{f}(z) = \int_0^\infty t^{z-1} f(t)\, dt.$$

Let $\partial = -t d/dt$, and for $a \in \tilde{\Sigma}^0_{\alpha,\beta}$, we define the Mellin operator $a(t, \partial) \in \operatorname{Op} \tilde{\Sigma}^0_{\alpha,\beta}$ by

(1–2) $$a(t, \partial) f(t) = \frac{1}{2\pi i} \int_{\operatorname{Re} z = \gamma} t^{-z} a(t, z) \tilde{f}(z)\, dz,$$

with $\gamma \in (\alpha, \beta)$.

If $f \in L^p(J)$ let Rf be the reflection

(1–3) $$Rf(t) = f(1 - t).$$

Definition 1.1. *An operator A from $C_0^\infty(J)$ to $C^\infty(J)$ is a Mellin operator in the class* $\operatorname{Op} \Sigma_{\alpha,\beta}(J)$ *iff*

(1) *For all $\phi, \psi \in C_0^\infty([0, 1))$, there are operators $a_{0\phi\psi}(t, \partial) \in \operatorname{Op} \tilde{\Sigma}^0_{\alpha,\beta}$ and $C_{0\phi\psi}$, compact on $L^p(J)$ for all p with $\frac{1}{p} \in (\alpha, \beta)$, such that*

(1–4) $$\phi A \psi = a_{0\phi\psi}(t, \partial) + C_{0\phi\psi}.$$

(2) *If $\phi, \psi \in C^\infty([0, 1])$ have disjoint supports, the operator $\phi A \psi$ is compact on $L^p(J)$, $\frac{1}{p} \in (\alpha, \beta)$.*

(3) *The operator $A^R \equiv RAR$ satisfies conditions* (1) *and* (2).

To define the *principal symbol*, $\mathrm{Smbl}^{\frac{1}{p}}(A)$, for A as an operator on $L^p(J)$, we use that there are uniquely defined functions $a_0(z)$, $a_{0\pm}(t)$ such that for all $\phi, \psi \in C_0^\infty([0, 1))$,

$$\text{(1-5)}\qquad \begin{aligned} a_{0\phi\psi}(0, z) &= \phi(0)a_0(z)\psi(0), && z \in \Gamma_{\alpha,\beta}, \\ a_{0\phi\psi}(t, \tfrac{1}{p} \pm i\infty) &= \phi(t)a_{0\pm}(t)\psi(t), && 0 \le t < 1,\ \tfrac{1}{p} \in (\alpha, \beta). \end{aligned}$$

There are uniquely defined functions $a_1(z)$, $a_{1\pm}(t)$ such that for all $\phi, \psi \in C_0^\infty([0, 1))$,

$$\text{(1-6)}\qquad \begin{aligned} (a^R)_{0\phi\psi}(0, z) &= \phi(0)a_1(z)\psi(0), && z \in \Gamma_{\alpha,\beta}, \\ (a^R)_{0\phi\psi}(t, \tfrac{1}{p} \pm i\infty) &= \phi(t)a_{1\pm}(t)\psi(t), && 0 \le t < 1,\ \tfrac{1}{p} \in (\alpha, \beta). \end{aligned}$$

Moreover

$$\text{(1-7)}\qquad a_{0\pm}(t) = a_{1\mp}(1 - t), \qquad 0 < t < 1.$$

Let $\mathscr{R}_J^{\frac{1}{p}}$ be the oriented boundary of the rectangle:

$$\text{(1-8)}\qquad \begin{array}{ccccc} & t=0 & t \in [0,1] & t=1 & \\ \frac{1}{p}+i\infty & \lceil & \text{———} & \rceil & \frac{1}{p}-i\infty \\ \Gamma_{\frac{1}{p}} & \uparrow & \mathscr{R}_J^{\frac{1}{p}} & \downarrow & \Gamma_{\frac{1}{p}} \\ \frac{1}{p}-i\infty & \lfloor & \text{———} & \rfloor & \frac{1}{p}+i\infty \\ & t=0 & t \in [0,1] & t=1 & \end{array}$$

Definition 1.2. *Let* $A \in \mathrm{Op}\,\Sigma_{\alpha,\beta}(J)$ *and* $\frac{1}{p} \in (\alpha, \beta)$. *The principal symbol of* A *as an operator on* $L^p(J)$, $\mathrm{Smbl}^{\frac{1}{p}}(A)$, *is the quadruple of functions* $a_0(\frac{1}{p}+i\xi)$, $a_{0+}(t) = a_{1-}(1-t)$, $a_1(\frac{1}{p}+i\xi)$, $a_{0-}(t) = a_{1+}(1-t)$, *considered as a continuous function on* $\mathscr{R}_J^{\frac{1}{p}}$:

$$\text{(1-9)}\qquad \begin{array}{ccccc} & t=0 & a_{0+}(t) = a_{1-}(1-t) & t=1 & \\ \frac{1}{p}+i\infty & \lceil & \text{———} & \rceil & \frac{1}{p}-i\infty \\ a_0(\frac{1}{p}+i\xi) & \uparrow & \mathscr{R}_J^{\frac{1}{p}} & \downarrow & a_1(\frac{1}{p}+i\xi) \\ \frac{1}{p}-i\infty & \lfloor & \text{———} & \rfloor & \frac{1}{p}+i\infty \\ & t=0 & a_{0-}(t) = a_{1+}(1-t) & t=1 & \end{array}$$

Definition 1.3. *Let* $A = (A_{ij})$ *be an* $N \times N$ *matrix of operators in* $\mathrm{Op}\,\Sigma_{\alpha,\beta}(J)$. *The system* A *is elliptic on* $L^p(J)$[2] *iff* $\mathrm{Smbl}^{\frac{1}{p}} A$ *is a nonsingular matrix on* $\mathscr{R}_J^{\frac{1}{p}}$. *A number* $z_0 \in \Gamma_{\alpha,\beta}$ *is a singularity type for* A *at* $t = 0$ $[t = 1]$ *if*

$$\text{(1-10)}\qquad \det(\mathrm{Smbl}^{\frac{1}{p}}(A)(0, z_0)) = 0 \quad [\det(\mathrm{Smbl}^{\frac{1}{p}}(A)(1, z_0)) = 0].$$

[2]For brevity we write $L^p(J)$ for $[L^p(J)]^N$.

The following is shown in [E, Theorems 4.4 and 4.6] and [LP, Theorems 4.1 and 4.2].

Theorem 1. *Let* $A = (A_{ij})$ *be an* $N \times N$ *matrix of operators in* $\mathrm{Op}\,\Sigma_{\alpha,\beta}(J)$. *Then*

(1) A *is a Fredholm operator on* $L^p(J)$ *iff* A *is elliptic on* $L^p(J)$.

(2) *If* A *is elliptic on* $L^p(J)$, *define*

$$\operatorname{ind}_p(A) = \dim((\ker A) \cap L^p(J)) - \dim((\ker A^*) \cap L^{p/p-1}(J)). \tag{1-11}$$

Then

$$\operatorname{ind}_p(A) = \frac{1}{2\pi} \Delta_{\mathscr{R}_J^{\frac{1}{p}}} \{\arg(\det(\mathrm{Smbl}^{\frac{1}{p}} A))\}, \tag{1-12}$$

where the change in arg *is taken as* $\mathscr{R}_J^{\frac{1}{p}}$ *is traversed in the clockwise direction.*

Remark. In treating boundary value problems in domains with corners it is useful to regard Mellin operators as acting on weighted spaces, e.g., $L^{p,\sigma}(J) \equiv \{f\colon t^\sigma f(t) \in L^p(J)\}$. In this case we suppose that both $\frac{1}{p} + \sigma$ and $\frac{1}{p}$ lie in (α, β). The *principal symbol* would be defined on the oriented rectangle $\mathscr{R}_J^{\frac{1}{p}+\sigma,\frac{1}{p}}$ whose left-hand side is the contour $\Gamma_{\frac{1}{p}+\sigma}$, and whose right-hand side is the contour $\Gamma_{\frac{1}{p}}$. Cf. [E], but note that our notation differs slightly from [E, (4.8) ff.]. The approach of weighted spaces is especially useful where different weights may be introduced at different vertices of a polygon.

When double layer potentials on a curvilinear polygon $\partial\Omega^+$ are reduced to a system of Mellin operators as in §3, the operators near $t = 1$ will correspond to a smooth part of $\partial\Omega^+$ so that singularities at $t = 1$ will not appear; the change in arg of $\det(\mathrm{Smbl}^{\frac{1}{p}} A)$ will occur entirely on the contour $\Gamma_{\frac{1}{p}}$ on the left-hand side of (1–8).

2. Examples of Mellin operators

In this section we give examples of Mellin operators in $\mathrm{Op}\,\Sigma_{0,1}(J)$.

1. The finite Hilbert transform H is defined by

$$Hf(t) = \text{p.v.}\, \frac{1}{\pi} \int_0^1 \frac{f(s)}{t-s}\, ds. \tag{2-1}$$

H is in $\operatorname{Op}\Sigma_{0,1}(J)$ and $\operatorname{Smbl}^{\frac{1}{p}} H$ is

$$\begin{array}{ccccc} & t=0 & +i & t=1 & \\ \frac{1}{p}+i\infty & & & & \frac{1}{p}-i\infty \\ -\cot\pi z & \uparrow & \mathscr{R}_J^{\frac{1}{p}} & \downarrow & +\cot\pi z \\ \frac{1}{p}-i\infty & & & & \frac{1}{p}+i\infty \\ & t=0 & -i & t=1 & \end{array} \tag{2-2}$$

2. Let $k(t)\in\mathscr{F}'_{-\infty,1}$ [LP, Definition 1.1]; i.e., $k(t)\in C^{\infty}([0,\infty))$ and for every $l\geq 0$, $\delta>0$, $\partial^l k(t)=O(t^{-1+\delta})$ as $t\to\infty$. Define the *Hardy kernel operator* by

$$Kf(t)=\int_0^1 k\left(\frac{t}{s}\right)f(s)\,\frac{ds}{s}. \tag{2-3}$$

Then $K\in\operatorname{Op}\Sigma_{0,1}(J)$ and $\operatorname{Smbl}^{\frac{1}{p}} K$ is

$$\begin{array}{ccccc} & t=0 & 0 & t=1 & \\ \frac{1}{p}+i\infty & & & & \frac{1}{p}-i\infty \\ \tilde{k}(z) & \uparrow & \mathscr{R}_J^{\frac{1}{p}} & \downarrow & 0 \\ \frac{1}{p}-i\infty & & & & \frac{1}{p}+i\infty \\ & t=0 & 0 & t=1 & \end{array} \tag{2-4}$$

Definition 2.1. *A function $k(x,y)$ is a double layer kernel if*

(1) $k\in C^{\infty}(\mathbf{R}^2\backslash\{0\})$,
(2) k *is homogeneous of degree* -1 *and odd: for all* $\lambda\neq 0$, $k(\lambda x,\lambda y)=\lambda^{-1}k(x,y)$.

3. Let $k(x,y)$ be a double layer kernel and $0<\theta<2\pi$. Define

$$K_\theta f(t)=\int_0^1 k(t-s\cos\theta,\,-s\sin\theta)f(s)\,ds. \tag{2-5}$$

Then K_θ is a Hardy kernel operator with kernel

$$k_\theta(t)=k(t-\cos\theta,\,-\sin\theta). \tag{2-6}$$

4. Let $k(x,y)$ be a double layer kernel. Then

$$\lim_{y\to 0^{\pm}}\int_0^1 k(t-s,\,y)f(s)\,ds=\pm c_k f(t)+\pi k(1,0)Hf(t), \tag{2-7}$$

where

$$c_k=\lim_{R\to\infty}\int_{-R}^{R}k(x,1)\,dx=\lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{\pi-\varepsilon}\frac{k(\cos\theta,\,\sin\theta)}{\sin\theta}\,d\theta. \tag{2-8}$$

This is simply the observation that if we let

$$\phi(t) = \begin{cases} k(t, 1) - k(1, 0)/t, & |t| > 1, \\ k(t, 1), & |t| < 1, \end{cases}$$

then $\phi(t) = O(1/t^2)$ as $|t| \to \infty$, so that $\phi \in L^1(\mathbf{R})$. The function

$$\frac{1}{y}\int_0^1 \phi(\frac{t-s}{y}) f(s)\, ds$$

is dominated by the Hardy-Littlewood maximal function of f and approaches $\pm(\int \phi(x)\, dx) f$ in $L^p(J)$ (cf. Stein [St]). Since $k(x, 0) = k(1, 0)/x$ is an odd function, $(\int \phi(x)\, dx)$ is given by (2–8).

5. Let $k(x, y)$ be a double layer kernel. Let $\vec{\gamma}_j$, $j = 1, 2$, be two C^∞ curves which intersect only at $(0, 0)$. Assume that $d\vec{\gamma}_j/dt\big|_{t=0} = \vec{u}_j$ are unit vectors, $\vec{u}_1 \neq \vec{u}_2$, so that $\vec{\gamma}_j(t) = t\vec{u}_j + \vec{\varepsilon}_j(t)$, with $\vec{\varepsilon}_j(t) = O(t^2)$. Let

$$K^{12} f(t) = \int_0^1 k(\vec{\gamma}_1(t) - \vec{\gamma}_2(s)) f(s) \left|\frac{d\vec{\gamma}_2}{ds}\right| ds. \tag{2–9}$$

Then K^{12} is a Mellin operator whose principal symbol is the same as that of the Hardy kernel operator with kernel

$$k^{12}(t) = k(t\vec{u}_1 - \vec{u}_2).$$

To show this we assume $\vec{u}_1 = (1, 0)$ and $\vec{u}_2 = (\cos\theta, \sin\theta)$, $0 < \theta < 2\pi$. Then $k(\vec{\gamma}_1(t) - \vec{\gamma}_2(s)) = k(t - s\cos\theta, -s\sin\theta) + R(t, s)$, where

$$R(t, s) = \int_0^1 \vec{\varepsilon}(t, s) \cdot \nabla k((t - s\cos\theta, -s\sin\theta) + \tau\vec{\varepsilon}(t, s))\, d\tau \tag{2–10}$$

with $\vec{\varepsilon}(t, s) = \vec{\varepsilon}(t) - \vec{\varepsilon}(s)$. Since $|\vec{\gamma}_1(t) - \vec{\gamma}_2(s)| \approx t + s$, we can differentiate wrt t to show that

$$f(t) \mapsto \frac{d}{dt}\int_0^1 R(t, s) f(s)\, ds$$

can be dominated by a Hardy kernel operator. Hence $f(t) \mapsto \int_0^1 R(t, s) f(s)\, ds$ is a compact operator on $L^p(J)$.

6. Let $\vec{\gamma}(t)$, $0 \le t \le 1$, be a C^∞ curve and $k(x, y)$ a double layer kernel. Let

$$K_{\vec{\gamma}} f(t) = \text{p.v.}\int_0^1 k(\vec{\gamma}(t) - \vec{\gamma}(s)) f(s) \left|\frac{d\vec{\gamma}}{ds}\right| ds. \tag{2–11}$$

Then $K_{\vec{\gamma}} \in \mathrm{Op}\, \Sigma_{0,1}(J)$ and has the same symbol as $\pi k(\vec{\gamma}'(t)) |d\vec{\gamma}/dt| H$. Observe that if $\vec{\gamma}(t) - \vec{\gamma}(s) = \vec{\gamma}'(t)(t)(t-s) + \vec{\varepsilon}(t, s)$, then

$$k(\vec{\gamma}(t) - \vec{\gamma}(s)) - \frac{k(\vec{\gamma}'(t))}{t-s} = \int_0^1 \vec{\varepsilon}(t, s) \cdot \nabla k(\vec{\gamma}'(t)(t-s) + \tau\vec{\varepsilon}(t, s))\, d\tau,$$

which gives rise to a compact operator on $L^p(J)$.

7. In Example 6 assume that $\vec{\gamma}$ is smooth for $-1 \le t \le +1$ and $d\vec{\gamma}(0)/dt = \vec{u}$. For $0 \le t \le 1$, let $\vec{\gamma}_1(t) = \vec{\gamma}(t)$, $\vec{\gamma}_2(t) = \vec{\gamma}(-t)$. The operator K^{12} of (2–9) has the same symbol as the Hardy kernel $k(\vec{u})\frac{1}{t+1}$. The kernel $s(t) = \frac{1}{\pi}\frac{1}{t+1}$ is the kernel for the Stieltjes transform and $\tilde{s}(z) = \csc \pi z$ [LP, (4.30)]. In particular, if we break a smooth curve $\vec{\gamma}(t)$, $-1 \le t \le 1$ at $t = 0$ the Hilbert transform $\text{p.v.} \int_{-1}^{+1} k(\vec{\gamma}(t) - \vec{\gamma}(s)) f(s) |d\vec{\gamma}/ds|\, ds$ is equivalent to the matrix of operators

$$K = \begin{pmatrix} H_{\vec{\gamma}_1} & K^{12} \\ K^{21} & H_{\vec{\gamma}_2} \end{pmatrix}, \tag{2–12}$$

which has principal symbol at $t = 0$ given by

$$\pi k(\vec{u}) \times \begin{pmatrix} -\cot \pi z & \csc \pi z \\ -\csc \pi z & \cot \pi z \end{pmatrix}. \tag{2–13}$$

Note that the characteristic polynomial of the matrix in (2–13) is $p(\lambda) = (\lambda + i)(\lambda - i)$.

3. Layer potentials on curvilinear polygons

Let Ω^+ be a simply connected[3] domain in $\mathbf{R}^2$ whose boundary is a simple closed curvilinear polygon. As $\partial\Omega^+$ is traversed in the counterclockwise direction label the successive N vertices as $P_2, P_4, \dots, P_{2N} = P_0$. Let $\overrightarrow{P_iP_j}$ be the oriented piece of $\partial\Omega^+$ between P_i and P_j. Suppose that $\overrightarrow{P_{2k}P_{2k+2}}$ is parametrized by $\vec{\gamma}(t)$, $0 \le t \le 2$. For $k = 1, \dots, N$, we introduce the false vertices $P_{2k-1} = \vec{\gamma}_{2k-2}(1)$ and then parametrize $\overrightarrow{P_{2k}P_{2k-1}}$ by $\vec{\gamma}_{2k-1}(t) \equiv \vec{\gamma}_{2k-2}(2 - t)$, $0 \le t \le 1$. When $t = 0$ each parametrization is at one of the original vertices; if $t = 1$, we are at a "midpoint". For $i = 1, \dots, 2N$, let θ_i be the angle interior to Ω^+ at P_i, $0 < \theta_i < 2\pi$; of course $\theta_{2k-1} = \pi$. We assume that at $t = 0, 1$, $d\vec{\gamma}_j/dt$ are unit vectors; the arclength on $\overrightarrow{P_iP_{i+1}}$ is given by $d\sigma = (-1)^i |d\vec{\gamma}_i/dt|\, dt$.

For f a scalar or vector function in $L^p(\partial\Omega^+)$, we define $f^i(t) = f(\vec{\gamma}_i(t))$, $0 \le t \le 1$, $i = 1, \dots, 2N$.

Assume that $c(x, y)$ is scalar or matrix function such that for each i, $i = 1, \dots, 2N$, $c^i(t) = c(\vec{\gamma}_i(t))$ is a smooth function. Let $k(x, y)$ be an odd double layer kernel. We define the *double layer potential*

$$Kf(P) = c(P)f(P) + \text{p.v.} \int_{\partial\Omega^+} k(P - Q) f(Q)\, d\sigma_Q. \tag{3–1}$$

Let

$$K^{i,j} f^j(t) = \delta_{i,j} c^j(t) f^j(t) + \text{p.v.} \int_0^1 k(\vec{\gamma}_i(t) - \vec{\gamma}_j(s)) f^j(s) (-1)^j \left| \frac{d\vec{\gamma}_j}{ds} \right| ds, \tag{3–2}$$

[3] If Ω^+ is multiply connected we apply the method to each component of $\partial\Omega^+$.

so that

$$(Kf)^i(t) = \sum_{j=1}^{2N} K^{i,j} f^j(t);$$

we write $\mathbf{K} = \left(K^{i,j}\right)_{i,j=1,\dots,2N}$ for the operator K interpreted as a big system of Mellin operators on $L^p(J)$.

Except in the cases $j = i-1, i, i+1 \pmod{2N}$, the operators $K^{i,j}$ have smooth kernels and thus are compact operators on $L^p(J)$. The operators $K^{2k,2k-1}$ and $K^{2k-1,2k}$ are Hardy kernel operators whose symbol is calculated by (2–7); in particular their principal symbol vanishes for $t > 0$. The operators $K^{2k,2k+1}$ and $K^{2k+1,2k}$ have principal symbol which vanishes for $0 < t < 1$; near $t = 1$, to calculate $\det(\mathrm{Smbl}^{\frac{1}{p}}(\mathbf{K}))$, we can apply an even number of row and column transpositions to reduce the symbol matrix to 2×2 block diagonal form. After applying the reflection (1–3), we are again reduced to considering the previous case at $t = 0$ with angle $\theta_{2k+1} = \pi$. The determinants of the matrix of principal symbols are summarized in Theorem 2.

Theorem 2. *For* $i = 1, \dots, 2N$, $\pmod{2N}$, *let* $K^{(i)}$ *denote the matrix of blocks*

$$K^{(i)} = \begin{pmatrix} K^{i-1,i-1} & K^{i-1,i} \\ K^{i,i-1} & K^{i,i} \end{pmatrix}. \tag{3–3}$$

Then at $t = 0$,

$$\det(\mathrm{Smbl}^{\frac{1}{p}}(\mathbf{K})) = \prod_{i=1}^{N} \det(\mathrm{Smbl}^{\frac{1}{p}}(K^{(2i)})). \tag{3–4}$$

At $t = 1$,

$$\det(\mathrm{Smbl}^{\frac{1}{p}}(\mathbf{K})) = \prod_{i=1}^{N} \det(\mathrm{Smbl}^{\frac{1}{p}}(K^{(2i-1)})). \tag{3–5}$$

At $z = \frac{1}{p} \pm i\infty$,

$$\det(\mathrm{Smbl}^{\frac{1}{p}}(\mathbf{K})) = \prod_{i=1}^{2N} \det(\mathrm{Smbl}^{\frac{1}{p}}(K^{i,i})). \tag{3–6}$$

4. Elastostatic double layer potentials in a plane sector

We give explicit calculations for the double layer potentials for the system of elastostatics and hydrostatics in a plane sector. In this section we fix θ, $0 < \theta < 2\pi$, and let Ω^+ be the sector of opening θ:

$$\Omega^+ = \{(x, y) : x = r\cos\phi,\ y = r\sin\phi,\ 0 < r < \infty,\ 0 < \phi < \theta\}. \tag{4–1}$$

Denote the two pieces of $\partial\Omega^+$ as $S_1 = \{(\tau, \rho) : \tau > 0, \rho = 0\}$ and $S_2 = \{(\tau, \rho) : \tau = l\cos\theta,\ \rho = l\sin\theta,\ l > 0\}$. We denote by $\vec{\nu}_1 = -\mathbf{j}$ and

$\vec{\nu}_2 = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$ the exterior normals to Ω^+ along S_1 and S_2. For a vector function $\mathbf{f} \in L^p(\partial\Omega^+)$, let $\mathbf{f}^1(t) = \mathbf{f}(t, 0)$, $\mathbf{f}^2(t) = \mathbf{f}(t\cos\theta, t\sin\theta)$.

For $(t, s) \notin \partial\Omega^+$, the double layer potential is defined as in (0–17):

$$\begin{aligned} \mathbf{u_T}(t, s) &= \int_{\partial\Omega^+} \mathbf{T}_{\vec{\nu}(\tau,\rho)}(\mathbf{\Gamma}(t-\tau, s-\rho))\mathbf{f}(\tau, \rho)\, d\sigma_{\tau,\rho} \\ &= \int_0^\infty \mathbf{T}_{\vec{\nu}_1}(\mathbf{\Gamma}(t-\tau, s))\mathbf{f}^1(s)\, d\tau \\ &\quad + \int_0^\infty \mathbf{T}_{\vec{\nu}_2}(\mathbf{\Gamma}(t - l\cos\theta, s - l\sin\theta))\mathbf{f}^2(l)(-1)\, dl. \end{aligned} \tag{4–2}$$

We have

$$\lim_{s\to 0^\pm} \mathbf{u_T}(t, s) = (\mathbf{u_T^\pm})^1(t) = \mathbf{K_T^{\pm 11}}\mathbf{f}^1(t) + \mathbf{K_T^{12}}\mathbf{f}^2(t), \tag{4–3}$$

where

$$\begin{aligned} \mathbf{K_T^{\pm 11}}\mathbf{f}^1(t) &= \pm\mathbf{I}\mathbf{f}^1(t) + \text{p.v.}\int_0^\infty \mathbf{T}_{\vec{\nu}_1(\tau,\rho)}(\mathbf{\Gamma}(t-\tau, 0))\mathbf{f}^1(\tau)\, d\tau, \\ \mathbf{K_T^{12}}\mathbf{f}^2(t) &= -\int_0^\infty \mathbf{T}_{\vec{\nu}_2(\tau,\rho)}(\mathbf{\Gamma}(t - l\cos\theta, s - l\sin\theta))\mathbf{f}^2(l)\, dl. \end{aligned} \tag{4–4}$$

The singular integral operators in $\mathbf{K_T^{\pm 11}}$ are multiples of the Hilbert transform by (2–6) and the operator $\mathbf{K_T^{12}}$ is a 2×2 matrix of Hardy kernel operators with $\text{Smbl}^{\frac{1}{p}}(\mathbf{K_T^{12}})$ near $t = 0$ given by the Mellin transform of the kernel. When the identity $\mathbf{I}$ and the Hilbert transform are considered as Mellin operators, their kernels are the distributions $\delta(t-1)$ and $h(t) = \text{p.v.}\,\frac{1}{\pi}\frac{1}{t-1}$ respectively.

For $(t, 0) \in S_1$ and $(\cos\theta, \sin\theta) \in S_2$, we define

$$d^2 = t^2 - 2t\cos\theta + 1 = (t-\cos\theta)^2 + \sin^2\theta. \tag{4–5}$$

For $j = 0, 1, 2, 3$, let

$$k_j(t) = \frac{1}{\pi}\frac{(t-\cos\theta)^j(\sin\theta)^{3-j}}{d^4}. \tag{4–6}$$

Let $\mathscr{E}(x, y)$ be one of the scalar kernels in the matrix fundamental solution (0–13). Then $k_{\mathscr{E}_\rho} = -\frac{\partial\mathscr{E}}{\partial y}$ and $k_{\mathscr{E}_\tau} = -\frac{\partial\mathscr{E}}{\partial x}$ are double layer kernels according to (Definition 2.1). We consider the following scalar double layer potentials:

$$\begin{aligned} u_{\mathscr{E}_\rho}(t, s) &= \int_{\partial\Omega^+} \frac{\partial}{\partial\rho}\{\mathscr{E}(t-\tau, s-\rho)\} f(\tau, \rho)\, d\sigma_{\tau,\rho}, \\ u_{\mathscr{E}_\tau}(t, s) &= \int_{\partial\Omega^+} \frac{\partial}{\partial\tau}\{\mathscr{E}(t-\tau, s-\rho)\} f(\tau, \rho)\, d\sigma_{\tau,\rho}. \end{aligned} \tag{4–9}$$

Taking limits as $s \to 0^\pm$, we obtain the following Mellin operators on $L^p(\mathbf{R}^+)$:

(4–10)

$$\begin{aligned} K_{\mathscr{E}_\rho}^{\pm 11} f^1(t) &= \lim_{s\to 0^\pm}\int_0^\infty -\frac{\partial\mathscr{E}}{\partial y}(t-\tau, s) f^1(\tau)\, d\tau &&= \int_0^\infty k_{\mathscr{E}_\rho}^{\pm 11}\left(\frac{t}{\tau}\right) f^1(\tau)\frac{d\tau}{\tau}, \\ K_{\mathscr{E}_\rho}^{12} f^2(t) &= \int_0^\infty -\frac{\partial\mathscr{E}}{\partial y}(t - l\cos\theta, -l\sin\theta) f^2(l)\, dl &&= \int_0^\infty k_{\mathscr{E}_\rho}^{12}\left(\frac{t}{l}\right) f^2(l)\frac{dl}{l}. \end{aligned}$$

Similarly, we obtain the operators $K_{\mathscr{E}_\tau}^{\pm 11}$ and $K_{\mathscr{E}_\tau}^{12}$ and their corresponding kernels $k_{\mathscr{E}_\tau}^{\pm 11}$ and $k_{\mathscr{E}_\tau}^{12}$. The Mellin kernels obtained are given in the following kernel list.

(4–11)

$\mathscr{E}(t-\tau, s-\rho)$	$k_{\mathscr{E}_\rho}^{\pm 11}$	$k_{\mathscr{E}_\tau}^{\pm 11}$	$k_{\mathscr{E}_\rho}^{12}$	$k_{\mathscr{E}_\tau}^{12}$
$\frac{1}{2\pi}\log((\tau-t)^2+(\rho-s)^2)$	$\mp\delta$	$-h$	k_0+k_2	$-k_1-k_3$
$\frac{1}{\pi}\frac{(\tau-t)(\rho-s)}{(\tau-t)^2+(\rho-s)^2}$	$-h$	0	k_1-k_3	k_0-k_2
$\frac{1}{\pi}\frac{(\tau-t)^2}{(\tau-t)^2+(\rho-s)^2}$	$\pm\delta$	0	$-2k_2$	$-2k_1$
$\frac{1}{\pi}\frac{(\rho-s)^2}{(\tau-t)^2+(\rho-s)^2}$	$\mp\delta$	0	$2k_2$	$2k_1$

In (4–11) we have used the notation δ and h for the distribution Mellin kernels $\delta(t-1)$ and p.v. $\frac{1}{\pi}\frac{1}{t-1}$ respectively.

To show the explicit dependence of the kernels on the parameter $b=\frac{\lambda+\mu}{\lambda+2\mu}$ (cf. (0–3)), we note the following "tricks" which follow from (0–3) and (0–14):

(4–12)

$$\mu m=\frac{b}{2},\qquad \mu n=1-\frac{b}{2},\qquad \mu(n+2m)=1+\frac{b}{2},\qquad \lambda(m-n)=1-2b,$$
$$\mu(2m-n)=\frac{3}{2}b-1,\qquad \mu(n-m)=1-b,\qquad \mu(n+m)=1.$$

We now give the structure of the operators $\mathbf{K}_{\mathbf{T}}^{\pm 11}$ and $\mathbf{K}_{\mathbf{N}}^{\pm 11}$.

Theorem 3. *Let*

$$\mathbf{K}_{\mathbf{T}^0}^{11}=\begin{pmatrix}0 & H\\ -H & 0\end{pmatrix}. \tag{4–13}$$

Then

$$\begin{aligned}\mathbf{K}_{\mathbf{T}}^{\pm 11}&=\pm\mathbf{I}+(1-b)\mathbf{K}_{\mathbf{T}^0}^{11},\\ \mathbf{K}_{\mathbf{N}}^{\pm 11}&=\pm\mathbf{I}+\frac{b}{2}\mathbf{K}_{\mathbf{T}^0}^{11}.\end{aligned} \tag{4–14}$$

Proof. With $\vec{\nu}=-\mathbf{j}$, we have that

$$\mathbf{T}_{\vec{\nu}}(\mathbf{u}(\tau,\rho))=-\begin{pmatrix}\mu u_{1,\rho}\\ \lambda u_{1,\tau}+(\lambda+2\mu)u_{2,\rho}\end{pmatrix}. \tag{4–15}$$

We apply $\mathbf{T}_{\vec{\nu}(\tau,\rho)}$ to the columns of the fundamental matrix $\mathbf{\Gamma}(t-\tau, s-\rho)$ and take limits as $s\to 0^{\pm}$. As a sample calculation we calculate the kernel in the 2, 1 position. Using the kernel list (4–11), we obtain

$$
\begin{aligned}
-k^{\pm 11}_{\mathbf{T},21} &= \lambda[n(-h) - m\cdot 0] + (\lambda+2\mu)[-m(-h)] \\
&= -h[\lambda n + (\lambda+2\mu)(-m)] \\
&= -h[\lambda(n-m) - 2\mu m] \\
&= -h[2b - 1 - 2\frac{b}{2}] \\
&= (1-b)h.
\end{aligned} \tag{4–16}
$$

Similarly

$$
-k^{\pm 11}_{\mathbf{N},21} = (\lambda+\mu)[n(-h) - m\cdot 0] + (\lambda+2\mu)[-m(-h)]. \tag{4–17}
$$

The method of simplification to be consistently applied is to collect the coefficients of λ and μ and then to use the tricks (4–12) to write the coefficients in terms of b.

The remaining very tedious calculations are left to the reader. □

To calculate the kernels in $\mathbf{K}^{12}_{\mathbf{T}}$ and $\mathbf{K}^{12}_{\mathbf{N}}$, we split the operators into

$$
\mathbf{K}^{12}_{\mathbf{T}} = \sin\theta \mathbf{K}_{\mathbf{T}_\mathbf{i}} - \cos\theta \mathbf{K}_{\mathbf{T}_\mathbf{j}},
$$

where

$$
\begin{aligned}
\mathbf{K}^{12}_{\mathbf{T}_\mathbf{i}}\mathbf{f}^2(t) &= \int_0^\infty \mathbf{T}_\mathbf{i}(\mathbf{\Gamma}(t - l\cos\theta, -l\sin\theta))\mathbf{f}^2(l)\,dl, \\
\mathbf{K}^{12}_{\mathbf{T}_\mathbf{j}}\mathbf{f}^2(t) &= \int_0^\infty \mathbf{T}_\mathbf{j}(\mathbf{\Gamma}(t - l\cos\theta, -l\sin\theta))\mathbf{f}^2(l)\,dl,
\end{aligned} \tag{4–18}
$$

and

$$
\begin{aligned}
\mathbf{K}^{12}_{\mathbf{N}_\mathbf{i}}\mathbf{f}^2(t) &= \int_0^\infty \mathbf{N}_\mathbf{i}(\mathbf{\Gamma}(t - l\cos\theta, -l\sin\theta))\mathbf{f}^2(l)\,dl, \\
\mathbf{K}^{12}_{\mathbf{N}_\mathbf{j}}\mathbf{f}^2(t) &= \int_0^\infty \mathbf{N}_\mathbf{j}(\mathbf{\Gamma}(t - l\cos\theta, -l\sin\theta))\mathbf{f}^2(l)\,dl.
\end{aligned} \tag{4–19}
$$

Note that the (-1) from the orientation has been omitted in the definitions (4–18) and (4–19).

Theorem 4. *The operators in* (4–18) *and* (4–19) *have the following structure*:

$$
\begin{aligned}
\mathbf{K}^{12}_{\mathbf{T}_\mathbf{i}} &= \mathbf{K}^{12}_{\mathbf{T}^0_\mathbf{i}} + b\mathbf{K}_{\mathbf{i}^b}, \quad \mathbf{K}^{12}_{\mathbf{T}_\mathbf{j}} = \mathbf{K}^{12}_{\mathbf{T}^0_\mathbf{j}} + b\mathbf{K}_{\mathbf{j}^b}, \\
\mathbf{K}^{12}_{\mathbf{N}_\mathbf{i}} &= \mathbf{K}^{12}_{\mathbf{N}^0_\mathbf{i}} + \frac{b}{2}\mathbf{K}_{\mathbf{i}^b}, \quad \mathbf{K}^{12}_{\mathbf{N}_\mathbf{i}} = \mathbf{K}^{12}_{\mathbf{N}^0_\mathbf{j}} + \frac{b}{2}\mathbf{K}_{\mathbf{j}^b},
\end{aligned} \tag{4–20}
$$

where the Hardy kernels are

$$(4\text{–}21)\qquad \begin{aligned}
\mathbf{K}^{12}_{\mathbf{T}^0_\mathbf{i}} &= \begin{pmatrix} -k_1 - k_3 & -k_0 - k_2 \\ k_0 + k_2 & -k_1 - k_3 \end{pmatrix}, \\
\mathbf{K}^{12}_{\mathbf{i}^b} &= \begin{pmatrix} k_1 - k_3 & k_0 + 3k_2 \\ -k_0 + k_2 & -k_1 + k_3 \end{pmatrix}, \\
\mathbf{K}^{12}_{\mathbf{N}^0_\mathbf{i}} &= \begin{pmatrix} -k_1 - k_3 & 0 \\ 0 & -k_1 - k_3 \end{pmatrix}, \\
\mathbf{K}^{12}_{\mathbf{T}^0_\mathbf{j}} &= \begin{pmatrix} k_0 + k_2 & -k_1 - k_3 \\ k_1 + k_3 & k_0 + k_2 \end{pmatrix}, \\
\mathbf{K}^{12}_{\mathbf{j}^b} &= \begin{pmatrix} -k_0 + k_2 & -k_1 + k_3 \\ -3k_1 - k_3 & k_0 - k_2 \end{pmatrix}, \\
\mathbf{K}^{12}_{\mathbf{N}^0_\mathbf{j}} &= \begin{pmatrix} k_0 + k_2 & 0 \\ 0 & k_0 + k_2 \end{pmatrix}.
\end{aligned}$$

Proof. A typical computation is for the kernel in the 1, 1 position.

$$(4\text{–}22)\qquad \begin{aligned} k^{12}_{\mathbf{T}_\mathbf{i},11} &= (\lambda + 2\mu)[n(-k_1 - k_3) - m(-2k_1)] + \lambda(-m)(k_1 - k_3) \\ &= k_1[(\lambda + 2\mu)(-n + 2m) - \lambda m] + k_3[(\lambda + 2\mu)(-n) + \lambda m]. \end{aligned}$$

To simplify the coefficients of k_1 and k_3, collect the coefficients of λ and μ, and apply the tricks (4–12) to obtain

$$k^{12}_{\mathbf{T}_\mathbf{i},11} = k_1(-1 + b) + k_3(-1 - b).$$

In calculating the remaining kernels, note that the coefficients to be calculated for $k^{12}_{\mathbf{T}_\mathbf{j},rs}$ are the negatives of the coefficients calculated for $k^{12}_{\mathbf{T}_\mathbf{i},sr}$.

Again the very tedious details are left to the reader. □

Taking into account the (-1) introduced by the orientation of the ray S_2, we have

$$(4\text{–}23)\qquad \begin{aligned} \mathbf{K}^{12}_{\mathbf{T}} &= \sin\theta \mathbf{K}^{12}_{\mathbf{T}_\mathbf{i}} - \cos\theta \mathbf{K}^{12}_{\mathbf{T}_\mathbf{j}}, \\ \mathbf{K}^{12}_{\mathbf{N}} &= \sin\theta \mathbf{K}^{12}_{\mathbf{N}_\mathbf{i}} - \cos\theta \mathbf{K}^{12}_{\mathbf{N}_\mathbf{j}}. \end{aligned}$$

We introduce

$$(4\text{–}24)\qquad \begin{aligned} \mathbf{K}^{12}_{\mathbf{T}^0} &= \sin\theta \mathbf{K}^{12}_{\mathbf{T}_\mathbf{i}^0} - \cos\theta \mathbf{K}^{12}_{\mathbf{T}_\mathbf{j}^0}, \\ \mathbf{K}^{12}_{\mathbf{N}^0} &= \sin\theta \mathbf{K}^{12}_{\mathbf{N}_\mathbf{i}^0} - \cos\theta \mathbf{K}^{12}_{\mathbf{N}_\mathbf{j}^0}, \\ \mathbf{K}^{12}_{\vec{\nu}^b} &= \sin\theta \mathbf{K}^{12}_{\mathbf{i}^b} - \cos\theta \mathbf{K}^{12}_{\mathbf{j}^b}, \end{aligned}$$

so that

$$(4\text{–}25)\qquad \begin{aligned} \mathbf{K}^{12}_{\mathbf{T}} &= \mathbf{K}^{12}_{\mathbf{T}^0} + b\mathbf{K}^{12}_{\vec{\nu}^b}, \\ \mathbf{K}^{12}_{\mathbf{N}} &= \mathbf{K}^{12}_{\mathbf{N}^0} + \frac{b}{2}\mathbf{K}^{12}_{\vec{\nu}^b}. \end{aligned}$$

Next we calculate $\mathbf{K}^{21}_{\{\cdot\}}$ and $\mathbf{K}^{22}_{\{\cdot\}}$.

Let U be the reflection about the ray $\{(t, s) = (l\cos\frac{\theta}{2}, l\sin\frac{\theta}{2}) : l > 0\}$:

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}. \tag{4–26}$$

Note that $UU = \mathbf{I}_2$ and that $\det U = -1$.

Then it is "obvious" geometrically or may be verified by a calculation that

$$\begin{aligned} \mathbf{K}_{\mathbf{T}}^{21} &= U\mathbf{K}_{\mathbf{T}}^{12}U, & \mathbf{K}_{\mathbf{T}}^{\pm 22} &= U\mathbf{K}_{\mathbf{T}}^{\pm 11}U, \\ \mathbf{K}_{\mathbf{N}}^{21} &= U\mathbf{K}_{\mathbf{N}}^{12}U, & \mathbf{K}_{\mathbf{N}}^{\pm 22} &= U\mathbf{K}_{\mathbf{N}}^{\pm 11}U. \end{aligned} \tag{4–27}$$

Hence both $\mathbf{K}_{\mathbf{T}}^{\pm}$ and $\mathbf{K}_{\mathbf{N}}^{\pm}$ have the structure

$$\mathbf{K}_{\{\cdot\}}^{\pm} = \begin{pmatrix} \mathbf{K}_{\{\cdot\}}^{\pm 11} & \mathbf{K}_{\{\cdot\}}^{12} \\ U\mathbf{K}_{\{\cdot\}}^{12}U & U\mathbf{K}_{\{\cdot\}}^{\pm 11}U \end{pmatrix}. \tag{4–28}$$

We let $\hat{U}$ be the 4×4 matrix

$$\hat{U} = \begin{pmatrix} \mathbf{I}_2 & 0 \\ 0 & U \end{pmatrix}. \tag{4–29}$$

Then

$$\hat{U}\mathbf{K}_{\{\cdot\}}^{\pm}\hat{U} = \begin{pmatrix} \mathbf{K}_{\{\cdot\}}^{\pm 11} & \mathbf{K}_{\{\cdot\}}^{12}U \\ \mathbf{K}_{\{\cdot\}}^{12}U & \mathbf{K}_{\{\cdot\}}^{\pm 11} \end{pmatrix}. \tag{4–30}$$

5. The symbols in a plane sector

We are now reduced to calculating the determinant of a matrix of Mellin symbols of the form

$$\mathrm{Smbl}^{\frac{1}{p}}(\hat{U}\mathbf{K}_{\{\cdot\}}^{\pm}\hat{U}) = \begin{pmatrix} \tilde{\mathbf{K}}_{\{\cdot\}}^{\pm 11} & \tilde{\mathbf{K}}_{\{\cdot\}}^{12}U \\ \tilde{\mathbf{K}}_{\{\cdot\}}^{12}U & \tilde{\mathbf{K}}_{\{\cdot\}}^{\pm 11} \end{pmatrix}. \tag{5–1}$$

First we note that if A and B are 2×2 matrices, then

$$\det\begin{pmatrix} A & B \\ B & A \end{pmatrix} = \det(A - B)\cdot\det(A + B). \tag{5–2}$$

Our goal is to express $\det(\tilde{\mathbf{K}}_{\{\cdot\}}^{\pm 11} \pm \tilde{\mathbf{K}}_{\{\cdot\}}^{12}U)$ as the difference of two squares so that the zeroes can easily be found.

We shall call *antireflective* a matrix of the form $C = \begin{pmatrix} c_{11} & c_{12} \\ -c_{12} & c_{11} \end{pmatrix}$; note that $\det C = c_{11}^2 + c_{12}^2$. We shall call *reflective* a matrix of the form $D = \begin{pmatrix} d_{11} & d_{12} \\ d_{12} & -d_{11} \end{pmatrix}$; note that $\det D = -(d_{11}^2 + d_{12}^2)$. Finally observe that if C is antireflective and D is reflective, then

$$\det(C \pm D) = (c_{11}^2 + c_{12}^2) - (d_{11}^2 + d_{12}^2) = \det C + \det D. \tag{5–3}$$

First we record the structure of $\mathrm{Smbl}^{\frac{1}{p}}(\mathbf{K}_{\{\cdot\}}^{\pm 11})$ near $t = 0$. If $\mathbf{K}_{\mathbf{T}^0}^{11}$ is as defined in (4–13), it is immediate that near $t = 0$,

$$\sin\pi z\ \mathrm{Smbl}^{\frac{1}{p}}(\mathbf{K}_{\mathbf{T}^0}^{11})(t, z) = \begin{pmatrix} 0 & -\cos\pi z \\ \cos\pi z & 0 \end{pmatrix}; \tag{5–4}$$

the matrix in (5–4) is antireflective.

Theorem 5. *Near* $t = 0$, *the matrices* $\mathrm{Smbl}^{\frac{1}{p}}(\mathbf{K}_{\{\cdot\}}^{\pm 11})$ *are antireflective; the symbols are given by*

$$\tag{5-5} \begin{aligned} \sin\pi z\, \mathrm{Smbl}^{\frac{1}{p}}(\mathbf{K}_{\mathbf{T}}^{\pm 11})(t,\, z) &= \begin{pmatrix} \pm\sin\pi z & -(1-b)\cos\pi z \\ (1-b)\cos\pi z & \pm\sin\pi z \end{pmatrix}, \\ \sin\pi z\, \mathrm{Smbl}^{\frac{1}{p}}(\mathbf{K}_{\mathbf{N}}^{\pm 11})(t,\, z) &= \begin{pmatrix} \pm\sin\pi z & -\frac{b}{2}\cos\pi z \\ \frac{b}{2}\cos\pi z & \pm\sin\pi z \end{pmatrix}. \end{aligned}$$

To calculate the symbols of the Hardy kernel operators in (4–21), we give the Mellin transforms of the kernels. First we introduce

$$\tag{5-6} \begin{aligned} C_\theta(z) &= \cos((\pi-\theta)z+\theta), \\ S_\theta(z) &= \sin((\pi-\theta)z+\theta). \end{aligned}$$

We list the following table of Mellin tranforms for the kernels $k_j(t)$ defined by (4–6):

$$\tag{5-7} \begin{aligned} \sin\pi z\, \tilde{k}_0(z) &= \tfrac{1}{2}\{(-z+2)\sin\theta C_\theta(z-1) - \cos\theta S_\theta(z-1)\}, \\ \sin\pi z\, \tilde{k}_1(z) &= -\tfrac{1}{2}\{(z-1)\sin\theta S_\theta(z-1)\}, \\ \sin\pi z\, \tilde{k}_2(z) &= \tfrac{1}{2}\{z\sin\theta C_\theta(z-1) - \cos\theta S_\theta(z-1)\}, \\ \sin\pi z\, \tilde{k}_3(z) &= \tfrac{1}{2}\{(z+1)\sin\theta S_\theta(z-1) + 2\cos\theta C_\theta(z-1)\}. \end{aligned}$$

For obvious reasons we note the following formulas which follow easily from (5–7) and the trigonometric addition formulas.

$$\tag{5-8} \begin{aligned} \sin\pi z(\tilde{k}_0(z) - \tilde{k}_2(z)) &= (-z+1)\sin\theta C_\theta(z-1), \\ \sin\pi z(\tilde{k}_1(z) - \tilde{k}_3(z)) &= -z\sin\theta S_\theta(z-1) - \cos\theta C_\theta(z-1), \\ \sin\pi z(\tilde{k}_0(z) + 3\tilde{k}_2(z)) &= (z+1)\sin\theta C_\theta(z-1) - 2\cos\theta S_\theta(z-1), \\ \sin\pi z(3\tilde{k}_1(z) - \tilde{k}_3(z)) &= (-z+2)\sin\theta S_\theta(z-1) + \cos\theta C_\theta(z-1), \\ \sin\pi z(\tilde{k}_0(z) + \tilde{k}_2(z)) &= \sin\theta C_\theta(z-1) - \cos\theta C_\theta(z-1) \\ &= -\sin((\pi-\theta)(z-1)), \\ \sin\pi z(\tilde{k}_1(z) + \tilde{k}_3(z)) &= \cos\theta C_\theta(z-1) + \sin\theta S_\theta(z-1) \\ &= \cos((\pi-\theta)(z-1)). \end{aligned}$$

The structure of the symbols of the operators (4–24) is explained in Theorem 6. We first introduce the reflective matrix

$$\tag{5-8} V = \begin{pmatrix} \sin\theta & -\cos\theta \\ -\cos\theta & -\sin\theta \end{pmatrix}.$$

Theorem 6. *The symbols of the operators* $\mathbf{K}_{\mathbf{T}^0}^{12}U$ *and* $\mathbf{K}_{\mathbf{N}^0}^{12}U$ *are reflective matrices and satisfy*

(5–10)

$$\begin{aligned} \sin\pi z\, \mathrm{Smbl}^{\frac{1}{p}}\, (\mathbf{K}_{\mathbf{T}^0}^{12}U)(t,\, z) &= \begin{pmatrix} \sin(\pi-\theta)(z-1) & -\cos(\pi-\theta)(z-1) \\ -\cos(\pi-\theta)(z-1) & -\sin(\pi-\theta)(z-1) \end{pmatrix}, \\ &= -\sin(\pi-\theta)zU - \cos(\pi-\theta)zV, \\ \sin\pi z\, \mathrm{Smbl}^{\frac{1}{p}}\, (\mathbf{K}_{\mathbf{N}^0}^{12}U)(t,\, z) &= -\sin(\pi-\theta)zU. \end{aligned}$$

The symbol of the operator $\mathbf{K}^{12}_{\vec{\nu}^b}$ *is a matrix of the form* $\{z \times$ *antireflective+reflective*$\}$ *and satisfies*

$$\sin \pi z \,\mathrm{Smbl}^{\frac{1}{p}} \left(\mathbf{K}^{12}_{\vec{\nu}^b} U\right)(t,\, z) = z \sin\theta \begin{pmatrix} \cos(\pi-\theta)z & -\sin(\pi-\theta)z \\ \sin(\pi-\theta)z & \cos(\pi-\theta)z \end{pmatrix} + \cos(\pi-\theta)zV. \tag{5-11}$$

Finally we are ready to calculate $\det\left(\tilde{\mathbf{K}}^{\pm 11}_{\{\cdot\}} \pm \tilde{\mathbf{K}}^{12}_{\{\cdot\}} U\right)$. To avoid further confusion, we now calculate $\det \mathrm{Smbl}^{\frac{1}{p}}\left(\mathbf{K}^{+}_{\{\cdot\}}\right)$.

Define

$$\begin{aligned} f^{\oplus\pm}_{\mathbf{T}}(z) &= \det\left(\sin \pi z(\tilde{\mathbf{K}}^{+11}_{\mathbf{T}} \pm \tilde{\mathbf{K}}^{12}_{\mathbf{T}} U)\right), \\ f^{\oplus\pm}_{\mathbf{N}}(z) &= \det\left(\sin \pi z(\tilde{\mathbf{K}}^{+11}_{\mathbf{N}} \pm \tilde{\mathbf{K}}^{12}_{\mathbf{N}} U)\right). \end{aligned} \tag{5-12}$$

Next define

$$\begin{aligned} g^{++}_{\mathbf{T}}(z) &= bz\sin\theta + (2-b)\sin(2\pi-\theta)z \\ &= -bz\sin(2\pi-\theta) + (2-b)\sin(2\pi-\theta)z, \\ g^{--}_{\mathbf{T}}(z) &= bz\sin\theta - (2-b)\sin(2\pi-\theta)z \\ &= -bz\sin(2\pi-\theta) - (2-b)\sin(2\pi-\theta)z, \\ g^{+-}_{\mathbf{T}} &= b(z\sin\theta + \sin\theta z), \\ g^{-+}_{\mathbf{T}} &= b(z\sin\theta - \sin\theta z). \end{aligned} \tag{5-13}$$

Let

$$\begin{aligned} g^{++}_{\mathbf{N}}(z) &= \frac{b}{2}z\sin\theta + \left(1-\frac{b}{2}\right)\sin(2\pi-\theta)z \\ &= -\frac{b}{2}z\sin(2\pi-\theta) + \left(1-\frac{b}{2}\right)\sin(2\pi-\theta)z, \\ g^{--}_{\mathbf{N}}(z) &= \frac{b}{2}z\sin\theta - \left(1-\frac{b}{2}\right)\sin(2\pi-\theta)z \\ &= -\frac{b}{2}z\sin\theta - \left(1-\frac{b}{2}\right)\sin(2\pi-\theta)z, \\ g^{+-}_{\mathbf{N}} &= \frac{b}{2}z\sin\theta + \left(1+\frac{b}{2}\right)\sin\theta z, \\ g^{-+}_{\mathbf{N}} &= \frac{b}{2}z\sin\theta - \left(1+\frac{b}{2}\right)\sin\theta z). \end{aligned} \tag{5-14}$$

Theorem 7. *We have that*

$$\begin{aligned} f^{\oplus\pm}_{\mathbf{T}}(z) &= g^{\pm+}_{\mathbf{T}}(z)\cdot g^{\pm-}_{\mathbf{T}}(z), \\ f^{\oplus\pm}_{\mathbf{N}}(z) &= g^{\pm+}_{\mathbf{N}}(z)\cdot g^{\pm-}_{\mathbf{N}}(z). \end{aligned} \tag{5-15}$$

Proof. Let

$$A^{\pm} = \sin \pi z(\tilde{\mathbf{K}}^{+11}_{\mathbf{T}} \pm \tilde{\mathbf{K}}^{12}_{\mathbf{T}} U). \tag{5-16}$$

Using (4–25), (5–10), and (5-11), the antireflective part of $A^{\pm}$ is

(5–17)

$$A^{\pm}_{\text{anti}} = \sin \pi z \big(\mathbf{I}_2 + (1-b)\tilde{\mathbf{K}}^{11}_{\mathbf{T}^0}\big) \pm z(\sin\theta) \begin{pmatrix} \cos(\pi-\theta)z & -\sin(\pi-\theta)z \\ \sin(\pi-\theta)z & \cos(\pi-\theta)z \end{pmatrix},$$

which has determinant given by

$$(5\text{–}18)\quad (\sin \pi z \pm bz \sin\theta \cos(\pi-\theta)z)^2 + ((1-b)\cos \pi z \pm bz \sin\theta \sin(\pi-\theta)z)^2.$$

From (4–25) and (5–11), the reflective part of $A^{\pm}$ is

$$(5\text{–}19)\qquad A^{\pm}_{\text{refl}} = \pm(\tilde{\mathbf{K}}^{12}_{\mathbf{T}^0} U + b\cos(\pi-\theta)z\, V),$$

which has determinant given by

$$(5\text{–}20)\qquad \begin{aligned} &-\Big[(\cos\theta \sin(\pi-\theta)z + (1-b)\sin\theta\cos(\pi-\theta)z)^2 \\ &\qquad + (\sin\theta\sin(\pi-\theta)z - (1-b)\cos\theta\cos(\pi-\theta)z)^2\Big] \\ &= -\big[\sin^2(\pi-\theta)z + (1-b)^2\cos^2(\pi-\theta)z\big]. \end{aligned}$$

Thus

(5–21)

$$\begin{aligned} f^{\oplus\pm}_{\mathbf{T}}(z) = {} & \{\sin^2\pi z - \sin^2(\pi-\theta)z\} + (1-b)^2\{\cos^2\pi z - \cos^2(\pi-\theta)z\} \\ & + b^2z^2\sin^2\theta \pm 2bz\sin\theta\{\sin\pi z\cos(\pi-\theta)z \\ & \qquad\qquad + (1-b)\cos\pi z\sin(\pi-\theta)z\}. \end{aligned}$$

In the last two terms of (5–21) we complete the square to obtain

(5–22)

$$f^{\oplus\pm}_{\mathbf{T}}(z) = \big(bz\sin\theta \pm (\sin\pi z\cos(\pi-\theta)z + (1-b)\cos\pi z\sin(\pi-\theta)z)\big)^2 + \text{rest},$$

where

$$(5\text{–}23)\qquad \begin{aligned} \text{rest} = {} & \sin^2\pi z - \sin^2(\pi-\theta)z + (1-b)^2\big[\cos^2\pi z - \cos^2(\pi-\theta)z\big] \\ & - \big(\sin\pi z\cos(\pi-\theta)z + (1-b)\cos\pi z\sin(\pi-\theta)z\big)^2 \\ = {} & -2(1-b)\sin\pi z\cos(\pi-\theta)z\cos\pi z\sin(\pi-\theta)z \\ & + \{\sin^2\pi z - \sin^2(\pi-\theta)z - \sin^2\pi z\cos^2(\pi-\theta)z\} \\ & + (1-b)^2\{\cos^2\pi z - \cos^2(\pi-\theta)z - \cos^2\pi z\sin^2(\pi-\theta)z\}. \end{aligned}$$

The two terms in $\{\cdot\}$ simplify respectively to $-\cos^2\pi z\sin^2(\pi-\theta)z$ and $-\sin^2\pi z\cos^2(\pi-\theta)z$ so that

$$(5\text{–}24)\qquad \text{rest} = -\{\cos\pi z\sin(\pi-\theta)z + (1-b)\sin\pi z\cos(\pi-\theta)z\}^2.$$

From (5–22) and (5–24), the function $f^{\oplus\pm}_{\mathbf{T}}$ has been written as the difference of two squares $\alpha^2 - \beta^2$ so that of course $f^{\oplus\pm}_{\mathbf{T}} = (\alpha+\beta)(\alpha-\beta)$. That the terms have the form given by (5–15) follows from the addition formulas.

The explicit calculations for $f^{\oplus\pm}_{\mathbf{N}}$ proceed in a like manner. □

Remark. In a similar manner we may calculate

$$(5\text{–}25)\qquad \begin{aligned} f_{\mathbf{T}}^{\ominus\pm}(z) &= \det\left(\sin\pi z(\tilde{\mathbf{K}}_{\mathbf{T}}^{-11} \pm \tilde{\mathbf{K}}_{\mathbf{T}}^{12}U)\right),\\ f_{\mathbf{N}}^{\ominus\pm}(z) &= \det\left(\sin\pi z(\tilde{\mathbf{K}}_{\mathbf{N}}^{-11} \pm \tilde{\mathbf{K}}_{\mathbf{N}}^{12}U)\right). \end{aligned}$$

In the calculation the determinant of the reflective part is unchanged and for the determinant of the antireflective part (5–18) is replaced by

$$(5\text{–}26)\quad (-\sin\pi z \pm bz\sin\theta\cos(\pi-\theta)z)^2 + ((1-b)\cos\pi z \pm bz\sin\theta\sin(\pi-\theta)z)^2.$$

The final result is that

(5–27)

$$\begin{aligned} \det\left(\sin\pi z\,\mathrm{Smbl}^{\frac{1}{p}}(\mathbf{K}_{\mathbf{T}}^{-})\right) &= (bz\sin\theta - b\sin(2\pi-\theta z))(bz\sin\theta - (2-b)\sin\theta z)\\ &\quad\times(bz\sin\theta + b\sin(2\pi-\theta z))(bz\sin\theta + (2-b)\sin\theta z). \end{aligned}$$

As expected, $\det\left(\sin\pi z\,\mathrm{Smbl}^{\frac{1}{p}}(\mathbf{K}_{\mathbf{T}}^{-})\right)$ has the same form as

$$\det\left(\sin\pi z\,\mathrm{Smbl}^{\frac{1}{p}}(\mathbf{K}_{\mathbf{T}}^{+})\right),$$

with the roles of θ and $2\pi-\theta$ interchanged, since $2\pi-\theta$ is the "interior" angle for the complement of Ω^+.

6. The singularities of the principal symbol

The zeroes and change in argument of $\det(\mathrm{Smbl}^{\frac{1}{p}}(\mathbf{K}_{\{\cdot\}}^{+})) = (\sin\pi z)^{-4} f_{\{\cdot\}}^{\oplus+}(z)\cdot f_{\{\cdot\}}^{\oplus-}(z)$ can be easily calculated from (5–15). Essentially we must consider functions of the form

$$(6\text{–}1)\qquad g_{\alpha,\gamma}(z) = \frac{\sin\gamma z}{\gamma z} - \alpha\frac{\sin\gamma}{\gamma},$$

where $-1 \le \alpha \le 1$ and $0 < \gamma < 2\pi$. An interesting discussion of all the complex zeroes of (6–1) is given in Vasilopoulos [V] or Karal and Karp [KK]. Let $g(Z) = \sin Z/Z$; of course $g(Z)$ has simple zeroes at $Z = \pm n\pi$, $n = 1, 2, \ldots$. The next lemma is a summary of the remarks of [V, pp. 57 ff.] and is proved using the Argument Principle.

Lemma 6.1. *Let* $0 < C < 1$. *Then the equation*

$$(6\text{–}2)\qquad g(Z) - C = 0$$

has exactly one root in the strip $\Gamma_{0,\pi}$, *has no roots in the strips* $\Gamma_{(2n-1)\pi,2n\pi}$, $n = 1, 2, \ldots$, *and has exactly two roots in the strips* $\Gamma_{2n\pi,(2n+1)\pi}$, $n = 1, 2, \ldots$.

The equation

$$(6\text{–}3)\qquad g(Z) + C = 0$$

has no roots in the strips $\Gamma_{(2n-2)\pi,(2n-1)\pi}$, $n = 1, 2, \ldots$, *and has exactly two roots in the strips* $\Gamma_{(2n-1)\pi,2n\pi}$, $n = 1, 2, \ldots$.

Proof. The lemma follows from calculating the change in argument of $g(Z)\pm C$ on the contours $\Gamma_{n\pi} = \{Z = n\pi + iY\colon -\infty < Y < +\infty\}$. Let

$$g_n(Y) = g(n\pi + iY) = (-1)^n\frac{(Y + n\pi i)\sinh(\pi Y)}{n^2\pi^2 + Y^2}.$$

The change in argument of $g_0(Y) \pm C$ is 0; the change in argument of $g_{2k-1}(Y) - C$ is 0 and the change in argument of $g_{2k}(Y) - C$ is -2π; in contrast, the change in argument of $g_{2k}(Y) + C$ is 0 and the change in argument of $g_{2k-1}(Y) + C$ has change in argument -2π. Taking into account the change in argument of $g(X \pm i\infty) \pm C$, the Argument Principle gives the lemma. □

We denote by γ_{crit} the point where the minimum value of $g(t)$ on $[0, 2\pi]$ occurs; $\tan \gamma_{\text{crit}} = \gamma_{\text{crit}}$; $\gamma_{\text{crit}} \approx 257°27'$.

Lemma 6.2. *Consider the equation*

$$g_{\alpha,\gamma}(z) = \frac{\sin \gamma z}{\gamma z} - \alpha \frac{\sin \gamma}{\gamma} = 0, \qquad z \in \Gamma_{0,1}. \tag{6–4}$$

(1) *Let* $\alpha = 1$. *For* $0 < \gamma \le \gamma_{\text{crit}}$, *the equation* (6–4) *has no roots in* $\Gamma_{0,1}$; *for* $\gamma_{\text{crit}} < \gamma < 2\pi$ *there is a single root* $z_0(1, \gamma) \in \Gamma_{0,1}$ *which decreases monotonically from* 1 *to* $\frac{1}{2}$ *as* γ *increases from* γ_{crit} *to* 2π.

(2) *Let* $-1 \le \alpha < 1$. *For* $0 < \gamma \le \pi$, *the equation* (6–4) *has no roots in* $\Gamma_{0,1}$; *for* $\pi < \gamma < 2\pi$ *there is a single root* $z_0(\alpha, \gamma) \in \Gamma_{0,1}$ *which, for fixed* α, *decreases monotonically from* 1 *to* $\frac{1}{2}$ *as* γ *increases from* π *to* 2π.

Proof. The stated roots are understood easily by sketching the graph of g on $[0, 2\pi]$. That there are no complex roots follows from Lemma 5.1. □

We are now ready to announce the zeroes of $\det(\text{Smbl}^{\frac{1}{p}}(\mathbf{K}_{\mathbf{T}}^{+}))$. First observe that if $b = 0$, we have that $g_{\mathbf{T}}^{+-}$ and $g_{\mathbf{T}}^{-+}$ are identically 0; in particular $\text{Smbl}^{\frac{1}{p}}(\mathbf{K}_{\mathbf{T}}^{+})(\frac{1}{p} \pm i\infty)$ has rank 2; this shows that the boundary operator $\mathbf{T}(\mathbf{u})\vec{\nu}$ does not cover L. The following theorem summarizes the roots of $\det(\text{Smbl}^{\frac{1}{p}}(\mathbf{K}_{\mathbf{T}}^{+})) = 0$ in $\Gamma_{0,1}$.

Theorem 8. (1) *For* $t = 0$:

$$\det(\text{Smbl}^{\frac{1}{p}}\, \mathbf{K}_{\{\cdot\}}^{+}) = \frac{1}{\sin^4 \pi z} g_{\{\cdot\}}^{++}(z) g_{\{\cdot\}}^{+-}(z) g_{\{\cdot\}}^{-+}(z) g_{\{\cdot\}}^{--}(z). \tag{6–5}$$

(2) *The equations* $g_{\mathbf{T}}^{++} = 0$ *and* $g_{\mathbf{N}}^{--} = 0$ *have roots where*

$$\frac{\sin(2\pi - \theta) z}{2\pi - \theta} = \frac{b}{2 - b} \frac{\sin(2\pi - \theta)}{2\pi - \theta}; \tag{6–6}$$

Equation (6–6) *has a root* z_0 *in* $\Gamma_{0,1}$ *for* $0 < \theta < \pi$ $(0 \le b < 1)$, *or for only* $0 < \theta < 2\pi - \gamma_{\text{crit}}$ $(b = 1)$.

(3) *The equations* $g_{\mathbf{T}}^{--} = 0$ *and* $g_{\mathbf{N}}^{++} = 0$ *have roots where*

$$\frac{\sin(2\pi - \theta) z}{(2\pi - \theta) z} = -\frac{b}{2 - b} \frac{\sin(2\pi - \theta)}{2\pi - \theta}. \tag{6–7}$$

Equation (6–7) *has a root* z_0 *in* $\Gamma_{0,1}$ *for* $0 < \theta < \pi$ $(0 \le b \le 1)$.

(4) *The equation* $g_{\mathbf{T}}^{+-} = 0$ *has a root where*

$$\frac{\sin\theta z}{\theta z} = -\frac{\sin\theta}{\theta}. \tag{6–8}$$

Equation (6–8) *has a root* z_0 *in* $\Gamma_{0,1}$ *iff* $\pi < \theta < 2\pi$.

(5) *The equation* $g_{\mathbf{T}}^{-+} = 0$ *has a root where*

$$\frac{\sin\theta z}{\theta z} = \frac{\sin\theta}{\theta}. \tag{6–9}$$

Equation (6–9) *has a root* z_0 *in* $\Gamma_{0,1}$ *iff* $2\pi - \gamma_{\mathrm{crit}} < \theta < 2\pi$.

(6) *The equation* $g_{\mathbf{N}}^{+-} = 0$ *has a root where*

$$\frac{\sin\theta z}{\theta z} = -\frac{b}{2+b}\frac{\sin\theta}{\theta}. \tag{6–10}$$

Equation (6–10) *has a root* z_0 *in* $\Gamma_{0,1}$ *iff* $\pi < \theta < 2\pi$.

(7) *The equation* $g_{\mathbf{N}}^{-+} = 0$ *has a root where*

$$\frac{\sin\theta z}{\theta z} = \frac{b}{2+b}\frac{\sin\theta}{\theta}. \tag{6–11}$$

Equation (6–11) *has a root* z_0 *in* $\Gamma_{0,1}$ *iff* $\pi < \theta < 2\pi$.

(8) *If* $0 < b \le 1$, *for* $0 < \frac{1}{p} \le \frac{1}{2}$ *the change in argument of* $\det(\mathrm{Smbl}^{\frac{1}{p}}\,\mathbf{K}_{\{\cdot\}}^{+})$ *on the contour* $\Gamma_{\frac{1}{p}}$ *is* 0.

(9) *If* $0 < b \le 1$, *when* $\theta = \pi$, *for* $0 < \frac{1}{p} < 1$ *the change in argument of* $\det(\mathrm{Smbl}^{\frac{1}{p}}\,\mathbf{K}_{\{\cdot\}}^{+})$ *on the contour* $\Gamma_{\frac{1}{p}}$ *is* 0.

Proof. Statement (1) is Theorem 6; statements (2)–(7) follow from Lemma 5.2. Statements (8) and (9) are proved by calculating the change in argument near $\frac{1}{p} = 0$ and the Argument Principle. □

Remark. At the zeroes of $\det(\mathrm{Smbl}^{\frac{1}{p}}\,\mathbf{K}_{\{\cdot\}}^{+})$ the eigenvectors of the the 2×2 matrices $A^{\pm}$ are easily computed; in turn the eigenvectors of $\hat{U}\tilde{\mathbf{K}}_{\{\cdot\}}^{+}\hat{U}$ and $\tilde{\mathbf{K}}_{\{\cdot\}}^{+}$ are calculated.

Definition 6.1. *With* $\mathbf{K}_{\{\cdot\}}^{\pm}$ *as in equation* (4–28), *for* $\frac{1}{p}$ *not a zero of* $\det(\sin\pi z\,\mathrm{Smbl}^{\frac{1}{p}}(\mathbf{K}_{\{\cdot\}}^{\pm}))$, *define*

(6–12)

$$I_{\{\cdot\}}^{\pm}(\tfrac{1}{p}, b, \theta) = [\textit{number of zeroes of } \det(\sin\pi z\,\mathrm{Smbl}^{\frac{1}{p}}(\mathbf{K}_{\{\cdot\}}^{\pm})) \textit{ in } (0, \tfrac{1}{p})].$$

We note the following facts about $I_{\{\cdot\}}^{\pm}(\frac{1}{p}, b, \theta)$.

(1) $I_{\{\cdot\}}^{\pm}(\frac{1}{p}, b, \theta) = \frac{1}{2\pi}$ (change in arg of $\det\tilde{\mathbf{K}}_{\{\cdot\}}^{\pm}$ on $\Gamma_{\frac{1}{p}}$).

(2) $I_{\{\cdot\}}^{+}(\frac{1}{p}, b, \theta) = I_{\{\cdot\}}^{-}(\frac{1}{p}, b, 2\pi - \theta)$.

(3) For $0 < \theta < \pi$, $I_{\mathbf{T}}^{+}(\frac{1}{p}, b, \theta) = I_{\mathbf{N}}^{+}(\frac{1}{p}, b, \theta)$.

(4) For $\pi < \theta < 2\pi$, $I_{\mathbf{T}}^{+}(\frac{1}{p}, b, \theta) = I_{\mathbf{T}}^{-}(\frac{1}{p}, b, 2\pi - \theta)$ is independent of b for $0 < b \le 1$.

Let us now return to the problem on the domain Ω^+ as described in §4. For $\mathbf{f} \in L^p(\partial\Omega^+)$, let

$$\mathbf{K}^{\pm}_{\mathbf{T}}\mathbf{f}(P) = \pm\mathbf{I}\mathbf{f}(P) + \text{p.v.}\int_{\partial\Omega^+} \mathbf{T}_{\vec{\nu}(Q)}(\mathbf{\Gamma}(X-Q))\mathbf{f}(Q)\, d\sigma_Q, \tag{6–13}$$

$$\mathbf{K}^{\pm}_{\mathbf{T}}\mathbf{f}(P) = \pm\mathbf{I}\mathbf{f}(P) + \text{p.v.}\int_{\partial\Omega^+} \mathbf{N}_{\vec{\nu}(Q)}(\mathbf{\Gamma}(X-Q))\mathbf{f}(Q)\, d\sigma_Q. \tag{6–14}$$

When (6–13) or (6–14) is written as a big $4N \times 4N$ system of Mellin operators as in (3–1) ff., the operators $K^{(2i)}$ of (3–3) correspond to the operator $\mathbf{K}^{\pm}_{\{\cdot\}}$ of (4–28) with $\theta = \theta_{2i}$; the operators $K^{(2i-1)}$ of (3–3) correspond to the operator $\mathbf{K}^{\pm}_{\{\cdot\}}$ of (4–28) with $\theta = \pi$. Using Theorem 2, Theorem 7, and Theorem 8, we obtain

Theorem 9. *Let* $\mathbf{K}^{\pm}_{\{\cdot\}}$ *denote one of the operators* (6–13) *or* (6– 14). *Then*

(1) *For* $1 < p < \infty$, $\mathbf{K}^{\pm}_{\{\cdot\}}$ *is a Fredholm operator on* $L^p(\partial\Omega^+)$ *iff for all* j, $j = 1, \dots, N$, *the operators* (4–28), *with* $\theta = \theta_{2j}$, *is a Fredholm operator on* $[L^p(\mathbf{R}^+)]^4$.
(2) *If* $b = 0$, $\mathbf{K}^{\pm}_{\mathbf{T}}$ *is not a Fredholm operator on* $L^p(\partial\Omega^+)$ *for any* p, $1 < p < \infty$.
(3) *If* $b = 0$, $\mathbf{K}^{\pm}_{\mathbf{N}}$ *is not a Fredholm operator on* $L^p(\partial\Omega^+)$ *iff for some* j, $j = 1, \dots, N$, $\sin(\theta_{2j}\frac{1}{p}) = 0$ *or* $\sin((2\pi - \theta_{2j})\frac{1}{p}) = 0$.
(4) *If* $0 < b \le 1$, $\mathbf{K}^{\pm}_{\{\cdot\}}$ *is a Fredholm operator on* $L^p(\partial\Omega^+)$ *for all* p, $2 \le p < \infty$.
(5) *If* $0 < b \le 1$, *the "bad values" of* p *in* $(1, 2)$, *for which the operators* $\mathbf{K}^{\pm}_{\{\cdot\}}$ *are not Fredholm on* $L^p(\partial\Omega^+)$ *form a discrete set of cardinality at most* $2N$.
(6) *If* p *is a "good value" for which* $\mathbf{K}^{\pm}_{\{\cdot\}}$ *is a Fredholm operator on* $L^p(\partial\Omega^+)$, *the index of* $\mathbf{K}^{\pm}_{\{\cdot\}}$ *on* $L^p(\partial\Omega^+)$ *is given by*

$$\operatorname{ind}_p(\mathbf{K}^{\pm}_{\{\cdot\}}) = \sum_{j=1}^{N} I^{\pm}_{\{\cdot\}}(\tfrac{1}{p}, b, \theta_{2j}). \tag{6–15}$$

Proof. The determinant of the symbols of (6–13) and (6–14) are calculated using Theorem 2. Statements (1), (2), and (3) follow from the formulas (5–13) and (5–14). Statements (4) and (5) follow from Theorem 2, statements (8) and (9), applied to the operators (4–28). Statement (6) is the Index Theorem, Theorem 1. □

Remarks. When uniqueness is shown for a double layer potential on $L^2(\partial\Omega^+)$, for the "good values" of p the index on $L^p(\partial\Omega^+)$ is the dimension of the kernel since uniqueness for the adjoint holds in $L^q(\partial\Omega^+)$, $2 \le q < \infty$.

In contrast to the case of a finite interval, for the "good values" of p, the operators (4–28) have index $= 0$ on $[L^p(\mathbf{R}^+)]^4$. Cf. [E] or [LP, Definition 3.2] for the correct notion of principal symbol in this case; the change in argument of $\det(\mathrm{Smbl}^{\frac{1}{p}}\, \mathbf{K}^{\pm}_{\{\cdot\}})$ at $t = 0$ is killed by the change in argument at $t = \infty$.

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